ELSEVIER

Physica A 311 (2002) 527–535

PHYSICA A

www.elsevier.com/locate/physa

Statistical physics of induced correlation in a simple market

David Sherrington, Esteban Moro *, Juan Pedro Garrahan

Department of Physics, University of Oxford, Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, UK

Received 15 March 2002

Abstract

A simple model for the collective behaviour of diverse speculative agents in a competitive market is considered from the point of view of statistical physics. The only information about other agents available to any one is the total trade effected at each time-step. Evidence is presented for correlated adaptation, phase transitions, scaling, regimes of non-equilibration and equilibration, and relevant stochasticity. An intermediate-level quasi-continuous micro-dynamics is derived and shown to have a novel character. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 02.50.Le; 05.65.+b; 05.70.Fh; 87.23.Ge

Keywords: Minority game; Econophysics; Disordered systems

1. Introduction

This paper reviews some recent work on the application of methodology of statistical physics to a simple but interesting model problem inspired by a stockmarket of many individual speculators attempting to profit by buying low and selling high but with only global macroscopic information on which to base decisions (and no knowledge of their fellow-speculators as individuals).

Before describing the model, one might ask why such an economics-based model should be of professional interest to statistical physicists. At two extremes one can consider the answer as a producer or a consumer, the former in terms of the potential of statistical physics to aid the economist by complementing conventional neo-classical

* Corresponding author. Departamento de Matematicas, Universidad Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganes, Spain. Tel.: +34-91-6249101; fax: +34-91-6249129.

E-mail address: emoro@math.uc3m.es (E. Moro).

PII: S0378-4371(02)00835-X

economics theory (which typically assumes a hyper-rationality which is often unrealistic and unhelpful), the latter in terms of the challenges that economics poses to the statistical physicist. Here we shall concentrate on the latter perspective, albeit remarking on some of the implications for the former. [1] Hence we note: (i) economics systems of many agents are many-body systems which typically show complex emergent cooperative behaviour even with simple rules of individual action, (ii) frustration, the key ingredient for complexity in material systems such as spin glasses, is integral to the operation of speculative stockmarkets, (iii) disorder, or non-uniformity of individual behaviour, is essential to efficient operation, (iv) as we shall see, there is evidence for phase transitions and important fluctuation phenomena, two of the elements of 'meat' in modern statistical physics, as well as fruitful stochasticity and non-equilibration, and (v) economics systems are usually non-Markovian and involve features not found in material systems, such as anticipating the future. There is thus both a potential for applying existing knowledge and also a challenge of new issues and features beyond those of conventional condensed matter.

2. The model

The model system we consider is one known as the minority game (MG) [1]. It is intended to mimic in a simple minimalist fashion a stockmarket of speculative agents bidding to profit by buying when the majority wish to sell (so that the price can be lowered) and selling when the majority wish to buy (so that a higher price can be negotiated). It comprises a large number of agents each of whom can act as buyer or seller, deciding on how to play at each time-step through the application of a personal strategy to commonly available information. Each agent has a small set of strategies, drawn randomly, independently and immutably with identical probabilities from a large suite. At each time-step each agent picks one of his or her strategies, based on points allocated cumulatively to the strategies according to their virtual performance in predicting the minority which actually occurred. For simplicity, no other rewards are given and no account is taken of friction (dealing costs and spreads).

The system has frustration in that rewards are for minority action and quenched disorder in the strategies allocated to each agent. There is no direct interaction between agents but correlation arises through the adaptive evolution of the strategy points.

The original formulation of the MG was based on a Boolean formulation in which each agent had only two possible choices, buy or sell with no weight attached to the order, and a straight minority determined the outcome. The strategies were Boolean functions and the information on which they acted was the minority choice for the previous m time-steps. Points were allocated to strategies based on their ability to predict the minority and strategy-use choices were deterministic. Simulations [2,3] clearly demonstrated correlated behaviour, an apparent phase transition and scaling behaviour. Here, however, we shall concentrate on an alternative version which allows for continuous-valued bids, stochastic decisions and continuous-valued point

[1] See also J.P. Garrahan, E. Moro and D. Sherrington, Quant. Finance 1 (2001) 246.

updates, and also incorporates the recognition [4] that random common information is equally effective in inducing effective correlation between agents as is the quasi-random sequence of actual past action.

Thus, in the continuous minority game [5] the common information on which all agents base their actions is taken at each time-step as a stochastically random uncorrelated noise vector $\vec{I}(t)$ of unit length in a D-dimensional space. The strategies $\vec{R}_i^\alpha$, where $i = 1, \ldots, N$ label the agents and $\alpha = 1, \ldots, s$ their corresponding strategies, are quenched vectors of length $\sqrt{D}$ chosen randomly, also in D-dimensional space. In the presence of information $\vec{I}(t)$, strategy $\vec{R}_i^\alpha$ yields a (real number) bid

$$b_i^\alpha(t) = \vec{R}_i^\alpha \cdot \vec{I}(t) . \tag{1}$$

However, at any time each agent employs only one of his or her strategies, which we denote by $\vec{R}_i^*(t)$, yielding an actual individual bid $b_i^*(t) = \vec{R}_i^*(t) \cdot \vec{I}(t)$ and a total bid

$$A(t) = \sum_i b_i^*(t) \equiv Na(t) . \tag{2}$$

Strategy choices are determined by points $P_i(t)$ which are updated to reward minority action by

$$P_i^\alpha(t+1) = P_i^\alpha(t) - b_i^\alpha a(t) ; \tag{3}$$

note that the $a(t)$ are determined by the actual strategies used at t but all P_i^α are updated. Since there is no bias towards positive or negative bid, the first relevant macroscopic observable is the normalized standard deviation σ of the total bid, or 'volatility', given with appropriate normalization by

$$\sigma^2 = N^{-1} \langle A^2 \rangle , \tag{4}$$

where $\langle \cdots \rangle$ refers to a temporal average. σ is self-averaging and so in theoretical analysis may be averaged over the actual strategy choice in the thermodynamic limit $N \to \infty$.

3. Simulations

Fig. 1 shows the results of simulation of the model [5] starting with all P_i^α zero. It shows all the same qualitative features as the original formulation, with now $d = D/N$ playing the role of relevant scaling dimension; i.e., the volatility at low d is higher than that for agents making purely random choices, $\sigma_r = 1$, decreasing monotonically to a minimum value below σ_r at a critical $d = d_c(s)$ and rising to approach σ_r asymptotically as $d \to \infty$. The deviation from σ_r is evidence for correlation between agents and the minimum at $d_c(s)$ suggests a phase transition. In fact, further probing reveals that d_c is a critical boundary between non-equilibrating and equilibrating behaviour, as is demonstrated in Figs. 2 and 3 for $s = 2$ (which Fig. 1 already shows has the crucial ingredients and to which we now specialise for simplicity). Fig. 2 shows that for $d < d_c$ the behaviour depends on the starting configuration, with the volatility always $< \sigma_r$ for systems started with each agent's points (arbitrarily) sufficiently biased in favour of one

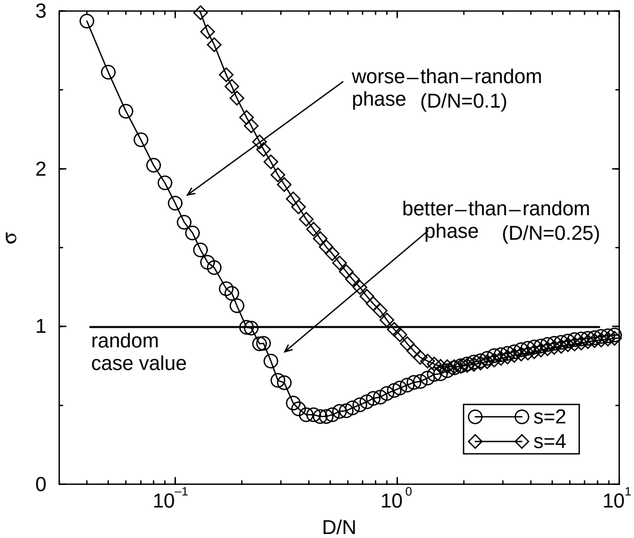

Fig. 1. Scaled volatility of the $T = 0$ minority game as a function of relevantly scaled information-strategy dimension, for $s = 2, 4$ and unbiased initial strategy weights (from Ref. [5]). The horizontal line is the corresponding value for uncorrelated behaviour.

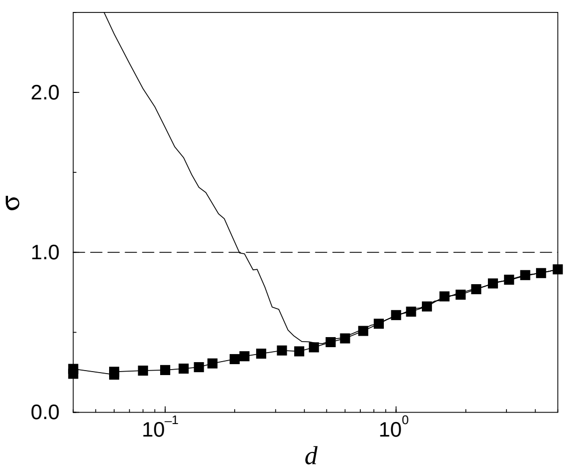

Fig. 2. Volatility for the $T = 0$ minority game $s = 2$ with randomly chosen biased initial strategy weights $|p_i(0)| \gg 0$ (□), compared with unbiased starts (—–).

of his or her strategies, in stark contrast to the case of unbiased starts (see Fig. 1) and thereby demonstrating non-equilibration. For $d > d_c$ the system equilibrates from any start. Fig. 3 shows a complementary 'hysteresis' behaviour in which, starting from an unbiased state at low d, the scaled dimension d is gradually increased quasi-statically into the region $d > d_c$, either by increasing D or decreasing N, and then quasi-statically

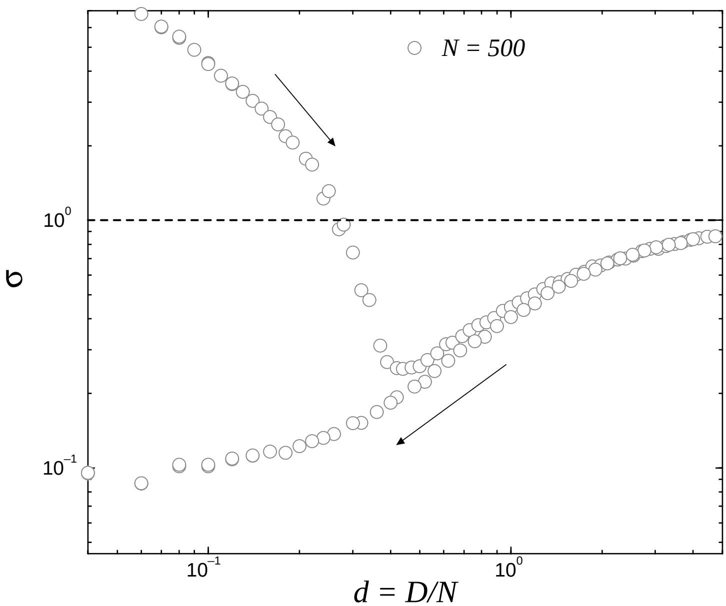

Fig. 3. Hysteresis effects in the volatility as d is quasi-statically evolved (increasing and decreasing slowly the parameter D for a given number of agents) from an unbiased point start. Numerically identical results are obtained by slow variation of N at fixed D.

reduced. In the region $d > d_c$ σ is identical in both directions, but for $d < d_c$ it enters the strongly biased regime of Fig. 2, again demonstrating non-equilibrium behaviour in this region.

A further interesting feature appears if one allows for stochasticity in agents' decisions. This is the case in the 'thermal minority game' (TMG) [5,6], where the strategy-use choices are probabilistic with a characteristic control measure analogous to a temperature. For $s = 2$ a single point measure per agent suffices

$$p_i(t) = P_i^1(t) - P_i^2(t) \tag{5}$$

and one can consider a strategy choice probability

$$\Pi_i^{1,2}(t) \sim \exp\{\pm f(p_i(t)/T\}\ . \tag{6}$$

The results of two choices are shown in Figs. 4(a) and (b), in each case for unbiased starts. The 'natural' choice of $f(p) = p$ was employed initially [5] and simulations demonstrated that for $d < d_c$ a non-zero T improves global performance in the sense of reducing σ. More particularly, a fairly sharp drop is observed to a minimum value less than both $\sigma(d = 0, T = 0)$ and σ_r at a temperature $T_c(d) \sim \mathcal{O}(1)$. The choice $f(p) = \text{sign}\, p$ shows essentially the same behaviour for T up to this critical value [6]. For $T > T_c$ the two choices differ however. This is a consequence of another feature, namely that above d_c, after an initial settling, the agents freeze with the $|p_i(t)|$ growing quasi-linearly with time. Consequently, for the initial choice of $f(p) = p$

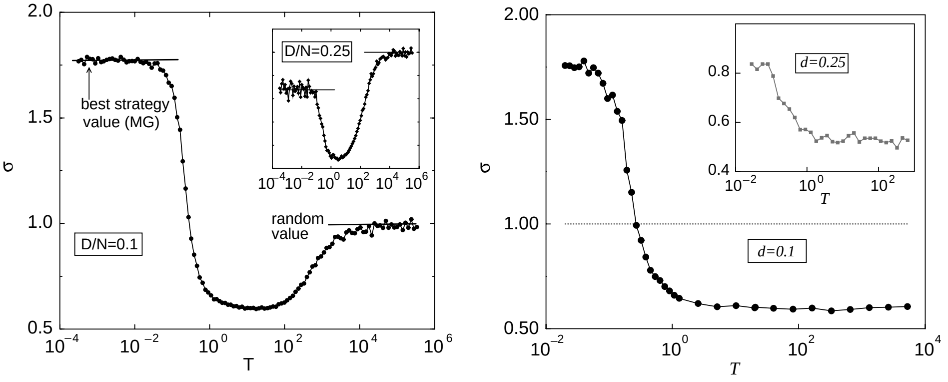

Fig. 4. Volatility as a function of T for (a) $f(p) = p$ (b) $f(p) = \text{sign}(p)$. In both cases $d < d_c$ and starts are unbiased.

time and temperature effectively cancel [7] so that in the long time limit σ does not change further with T. However, interestingly and potentially valuably, it stays at its minimum value [8]. By contrast, for $f(p)=$ sign(p) σ moves asymptotically towards σ_r as T is increased further beyond T_c towards ∞ [6], as expected. Both choices show that stochastic 'irrationality' can improve performance, demonstrate a thermal phase transition and extrapolate to a phase boundary in (d,T) space separating the non-equilibrating (low d and low T) region from the equilibrating one (high d or high T) with minimum volatility on the boundary.

4. Theory

Let us now turn to theory and consider the derivation of a coarse-grained quasi-continuous microdynamics, as an intermediate step towards an analytic derivation both of the macro-dynamics and of quasi-equilibrium in the relevant regime of its applicability. Again for simplicity, without losing conceptual generality, we take $s = 2$. It is also convenient to re-define parameters [7,6]

$$\vec{\omega}_i \equiv (\vec{R}_i^1 + \vec{R}_i^2)/2, \quad \vec{\xi}_i \equiv (\vec{R}_i^1 - \vec{R}_i^2)/2, \quad s_i(t) \equiv \Pi_i^1(t) - \Pi_i^2(t) \tag{7}$$

so that the choice of strategies used at each stage is given by

$$\vec{R}_i^*(t) = \vec{\omega}_i + \vec{\xi}_i \,\text{sign}\,(s_i(t) + \mu(t)), \tag{8}$$

where $\mu(t)$ is a stochastic random variable uniformly distributed between -1 and $+1$ and independently distributed in time. The equations for the point differences then read

$$p_i(t+1) = p_i(t) - (\vec{a}(t) \cdot \vec{I}(t))(\vec{\xi}_i \cdot \vec{I}(t)), \tag{9}$$

where $\vec{a}(t) \equiv \sum \vec{R}_i^*(t)/N$. Together with the random processes for $\vec{I}(t)$ and $\vec{R}_i^*(t)$, Eqs. (7) define the microdynamics of the TMG.

It is useful to consider the system on quasi-continuum coarse-grained mesoscopic time-scales in such a way as to simplify but preserve all the macroscopic features of the TMG. To this end we average over the random processes $\vec{I}(t)$ to yield an effective interaction between the agents [3,6,9]

$$\Delta p_i = -(ND)^{-1}\sum_j \vec{R}_j^*(t) \cdot \vec{\xi}_i \Delta t + \mathcal{O}(\Delta t^2)\,. \tag{10}$$

At $T = 0$ these equations are deterministic and

$$\Delta p_i = -\left[h_i + \sum_j J_{ij}\,\mathrm{sign}\,(p_j(t))\right]\Delta t + \mathcal{O}(\Delta t^2)\,, \tag{11}$$

where

$$h_i = (ND)^{-1}\sum_j \vec{\omega}_j \cdot \vec{\xi}_i, \quad J_{ij} = (ND)^{-1}\vec{\xi}_i \cdot \vec{\xi}_j\,, \tag{12}$$

while the volatility is given by

$$\sigma^2 = \Omega + 2\sum_i \overline{h_i\langle \mathrm{sign}(p_i(t))\rangle} + \sum_{ij}\overline{J_{ij}\langle \mathrm{sign}(p_i(t))\,\mathrm{sign}(p_j(t))\rangle}\,. \tag{13}$$

Simulation of Eq. (11) demonstrates that it correctly reproduces the behaviour of the original model, including the non-equilibrium preparation-dependence for $d < d_c$.

At finite temperature the stochastic choice of strategies is relevant and fluctuation effects cannot be ignored [6]. The transition probabilities in the large N limit are given by

$$W(\boldsymbol{p}'|\boldsymbol{p}) = \Phi(\nabla_{\boldsymbol{s}}\mathcal{H};\mathcal{M})\,, \tag{14}$$

where Φ corresponds to a normal distribution with mean $\nabla_{\boldsymbol{s}}\mathcal{H}$, where

$$\mathcal{H}[\boldsymbol{p}(t)] = \frac{1}{2}\,\Omega + \sum_i h_i s_i + \frac{1}{2}\sum_{ij} J_{ij}s_i s_j \tag{15}$$

and covariance matrix $\mathcal{M} = \{M_{ij}\}$ with

$$M_{ij}[\boldsymbol{p}(t)] = \sum_k J_{ik}J_{jk}(1 - s_k^2(t))\,. \tag{16}$$

This then yields the Fokker–Planck equation for the point (or strategy probability) distribution

$$\frac{\partial p}{\partial t} = -\sum_i \frac{\partial}{\partial p_i}\left(\frac{\partial \mathcal{H}}{\partial s_i}P\right) + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial p_i \partial p_j}(M_{ij}P) \tag{17}$$

and the effective stochastic differential equation for the point differences

$$\mathrm{d}\boldsymbol{p} = -\Delta_s\mathcal{H}\,\mathrm{d}t + \mathcal{M}\cdot\mathrm{d}\boldsymbol{W}\,, \tag{18}$$

where $\boldsymbol{W}$ is an N-dimensional Wiener process. The volatility is given by

$$\sigma^2 = 2\overline{\langle \mathscr{H} \rangle} + \sum_i \overline{J_{ii}(1 - \langle s_i^2 \rangle)} \, . \tag{19}$$

Again, numerical iteration of Eq. (18) confirms that it reproduces the results of simulation of the original TMG.

5. Interpretations

Let us now turn to interpretation. Eq. (16) is suggestive of a descent in an energy landscape but it should be noted that the gradient is with respect to s, not p, so that a metric is necessary, and at finite T there is also the unusual extra diffusive/noise term $\mathscr{M} \cdot \mathrm{d}\boldsymbol{W}$. Notwithstanding these subtleties $\mathscr{H}$ can be interpreted as a combination of an exchange between spins together with a random field term. Furthermore, the exchange J_{ij} has a form reminiscent of the Hopfield model for neural networks with the Cartesian coordinates of the ξ analogous to the different memories of the Hopfield model. However, it differs in the important regard of having the opposite sign, hindering rather than assisting freeze-out/retrieval into one of the Cartesian directions/patterns. Nor does it favour a simple exchange-induced 'spin-glass' like correlation over a 'paramagnet' as becomes evident if one chooses the agents' strategies to be equal and opposite, $\vec{R}_i^2 = -\vec{R}_i^1$, so as to eliminate the random field terms h_i; in this case the volatility never drops below σ_r, as demonstrated in Fig. 4. Indeed, we deduce that it is the random field terms that lead to the better-than-average volatility of the original models. Minimization of H, using the techniques of spin glass theory developed for neural networks, yields results in accord with simulation at $T = 0$ above the critical d_c both for fully random strategies [7] and also for the case of $\vec{R}_i^2 = -\vec{R}_i^1$ with $\vec{\boldsymbol{R}}_i^1$ random, and give d_c as a validity breakdown.

Finally, we note that the origin of the difference of behaviour between $d \gg d_c$ and $d \ll d_c$ can be attributed to the fact that in the former case $\vec{\boldsymbol{R}}$ is dilutely distributed on the hypersphere while in the latter it is densely distributed with large clusters. Indeed this is analogous to Johnson's picture of crowds and anticrowds [10] with the overlap of collective $\vec{R}_i^*$ with Cartesian directions in strategy space

$$n_\mu(t) = D^{-1/2} \sum_i \vec{R}_i^*(t) \cdot \hat{e}^\mu \, , \tag{20}$$

where $\hat{e}^\mu$ is a unit vector in the μth direction, corresponding to the difference of size between a crowd and an anticrowd [10] and

$$\sigma^2 = N^{-1} \sum_\mu \overline{n_\mu(t)^2} \, . \tag{21}$$

Notice that large fluctuations in the region $d < d_c$ are a consequence of period-two dynamical anticorrelations between crowds and anticrowds [9]. Temperature-induced stochasticity effectively loosens these anticorrelated clusters, while, even at $T = 0$, zero time preferences prevent the initial creation of these clusters.

6. Conclusion

With a very simple model we have seen that markets provide challenges and opportunities for statistical physics. Already this minimalist model is subtle and demonstrates non-Markovian behaviour, novel dynamics, phase transitions, non-equilibrium and fruitful irrationality. Although the system is mean field-like it is far from trivial.

Acknowledgements

The authors thank Andrea Cavagna and Irene Giardina for their collaboration in part of the work reviewed here and Matteo Marsili for discussions. Support is acknowledged from EC Grant ARG/B7-3011/94/27, EPSRC Grant GR/MO4426 and the Glasstone Research Fund. We also acknowledge benefit from ESF programme SPHINX.

References

[1] D. Challet, Y.-C. Zhang, Physica A 246 (1997) 407.
[2] R. Savit, R. Manuca, R. Riolo, Phys. Rev. Lett. 82 (1999) 2203.
[3] R. Challet, M. Marsili, Phys. Rev. E 60 (1999) R6271.
[4] A. Cavagna, Phys. Rev. E 59 (1999) R3783.
[5] A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. 83 (1999) 4429.
[6] J.P. Garrahan, E. Moro, D. Sherrington, Phys. Rev. E 62 (2000) R9.
[7] D. Challet, M. Marsili, R. Zecchina, Phys. Rev. Lett. 85 (2000) 5008.
[8] A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. 85 (2000) 5009.
[9] J.A.F. Heimel, A.C.C. Coolen, Phys. Rev. E 63 (2001) 56121.
[10] N.F. Johnson, M. Hart, P.M. Hui, Physica A 269 (1999) 1.