Statistical physics of adaptive correlation of agents in a market

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Abstract. Recent results and interpretations are presented for the thermal minority game, concentrating on deriving and justifying the fundamental stochastic differential equation for the microdynamics.

Market economics poses several problems of potential interest and challenge to statistical physics, involving the co-operative behaviour of many agents whose actions involve mutual frustration and disorder, both quenched and stochastic. In a nutshell, speculators in an idealized stock market are made up of buyers and sellers, each having personal gain as their objectives, trying to buy low and sell high, making their decisions based on commonly available information using individual strategies, with their collective actions determining the (time-varying) 'right choices' and learning from experience. From the point of view of the market regulator, however, preference is for low volatility and market efficiency.

The *minority game* (MG) is a simple encapsulation of some of the ingredients and issues of a market. It consists of N agents each of whom at each step of a parallel dynamical process makes either of two choices, with the objective of being in the minority overall. The agents have no direct knowledge of one another and make their decisions based on purely global information $\vec{I}(t)$, available equally to all. Their decisions are determined through the application to $\vec{I}(t)$ of individual strategy functions, each agent having a small number of such strategies, drawn randomly and independently from a large distribution at the outset and fixed throughout the game. At each time-step each agent employs (just) one of his or her strategies. Adaptation occurs through the development of functions which determine their choices of strategy.

In the original formulation [1] the information $\vec{I}(t)$ was the minority choice over the last m time-steps and the adaptation was achieved through the cumulative award of points at each time-step to the strategies which would have yielded the actual minority choice at that step. The strategy played by any agent at any time-step was that of his/her strategies which currently had the largest point-score.

A remarkable observation in simulations [2] was that the variance in the minority

choice became smaller than that of random choice for large enough m, indicating correlation of the agents' actions. A critical memory length m_c was observed for minimum variance, with agents appearing to be frozen in their choices for $m > m_c$, non-frozen for $m < m_c$. Moreover, it was shown that the dependence on m was through the scaling variable D/N, where $D = 2^m$ was the dimension of the space of strategies [2]. Further simulations showed (i) these results are unaffected by replacing the true history by a random $\vec{I}(t)$ [3], indicating that as far as macroscopic observables are concerned the 'information' merely effectuates the correlation; (ii) replacing the deterministic strategy-choices by stochastic ones can significantly reduce the volatility for information vectors of less than the critical length [4].

Here we consider the determination of a fundamental analytic theory and report the derivation of the underlying stochastic differential equation for the microdynamics [5]. We concentrate on a continuous formulation in which $\vec{I}(t)$ is a stochastically randomly chosen unit-length vector on a D-dimensional hypersphere, the strategies are quenched random vectors of length $\sqrt{D}$ in the same space, $\vec{R}_i^\alpha, i = 1, \ldots, N$ labeling the agents and the $\alpha = 1, \ldots, s$ their strategies. The analogues of the binary choices above are bids $b_i^\alpha(t) = \vec{R}_i^\alpha \cdot \vec{I}(t)$. The strategies which are actually used are indicated by $\vec{R}_i^*(t)$. The total bid at time t is $A(t) = \sum_i \vec{R}_i^*(t) \cdot \vec{I}(t)$. The point update rule is

$$P_i^\alpha(t+1) = P_i^\alpha(t) - b_i^\alpha(t)A(t)/N. \tag{1}$$

For simplicity we specialize to $s = 2$ and define

$$\vec{\xi}_i \equiv (\vec{R}_i^1 - \vec{R}_i^2)/2, \quad \vec{\omega}_i \equiv (\vec{R}_i^1 + \vec{R}_i^2)/2; \quad p_i(t) = P_i^1(t) - P_i^2(t) \tag{2}$$

In a generalized *thermal minority game* (TMG) the probability of strategy use is

$$\pi_i^{1,2}(t) \equiv [1 + \exp(\mp\beta f(p_i(t)))]^{-1} \tag{3}$$

and it is useful to define a 'spin'

$$s_i(t) \equiv \pi_i^1(t) - \pi_i^2(t) = \tanh(\beta f(p_i(t))). \tag{4}$$

In [4] the choice $f(p) = p$ was employed, but here we consider $f(p) = \mathrm{sgn(p)} \equiv \mathrm{z}$ [5].

We are interested in coarse-grained average behaviour on a time-scale greater than the step-length in order to pass to a continuum-time theory. Equivalently, we take a time-scale Δt with $\vec{I}(t)$ a differential random noise $\vec{I}(t) = \Delta\vec{W}(t)$ with zero mean and variance Δt. In the limit $\Delta t \to \infty$ a Kramers-Moyal expansion yields [5]

$$dp_i(t) = -(ND)^{-1}\vec{R}_i^*(t) \cdot \vec{\xi}_i dt + \mathcal{O}(dt^2) \tag{5}$$

so that to $\mathcal{O}(dt)$ the information noise has been eliminated in favour of an effective interaction between the agents and the averaged variance becomes

$$\sigma^2 \equiv N^{-1}\overline{\langle A(t)^2\rangle} = (ND)^{-1}\sum_{ij}\langle R_i^*(t)\cdot R_j^*(t)\rangle, \tag{6}$$

where the $\langle\cdot\rangle$ refer to a temporal average and the bar to an average over the quenched disorder of the strategies.

At $T = 0$ Eqs. (2) are deterministic and to leading order in dt reduce to

$$d\boldsymbol{p}/dt = -\boldsymbol{\nabla}_{\boldsymbol{s}}\mathcal{H}|_{\boldsymbol{s}=\boldsymbol{z}}; \qquad \boldsymbol{p} \equiv (p_1, ...p_N) \tag{7}$$

$$\mathcal{H} = \sum_i h_i s_i + \frac{1}{2}\sum_{i\neq j} J_{ij}s_i s_j, \tag{8}$$

$$h_i = (ND)^{-1}\sum_j \vec{\omega}_j\cdot\vec{\xi}_i, \quad J_{ij} = (ND)^{-1}\vec{\xi}_i\cdot\vec{\xi}_j \tag{9}$$

At finite temperature correlations between the fluctuations of the right hand sides of Eqs. (5) are of the same order as the mean and Eq. (7) must be replaced by a set of stochastic differential equations [5]

$$d\boldsymbol{p} = -\boldsymbol{\nabla}_{\boldsymbol{s}}\mathcal{H}dt + \mathcal{M}\cdot d\boldsymbol{W} \tag{10}$$

where $\mathcal{M} \equiv \{M_{ij}\}$ is the covariance matrix

$$M_{ij}[\boldsymbol{p}(t)] = \sum_\kappa J_{ik}J_{jk}[1 - s_k^2(t)] \tag{11}$$

and $\boldsymbol{W}(t)$ is an N-dimensional Wiener process of unit scale; for $T = 0 \quad s_k(t)^2 = 1$ so the Wiener term has no weight. Correspondingly, the Fokker-Planck equation for the probability distribution of the p_i is

$$\frac{\partial P}{\partial t} = -\sum_i \frac{\partial}{\partial p_i}\left(\frac{\partial\mathcal{H}}{\partial s_i}P\right) + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial p_i\partial p_j}(M_{ij}P). \tag{12}$$

The average volatility is given by

$$\sigma^2 \equiv N^{-1}\overline{\langle A(t)^2\rangle} = 1 + 2\overline{\langle\mathcal{H}\rangle} \tag{13}$$

Eq. (10) is thus the fundamental microscopic equation from which the macrodynamics should be calculable. To check this we have compared numerical evaluations of the volatility and the density of frozen agents (those for whom $p_i(t)$ does not change sign after initial transient) from Eqs. (7) and (10) with corresponding direct simulations from Eqs. (1) and (3). They are in perfect accord. This is shown explicitly in Fig. 1 for $T = 0$. Figs. 2a and 2b show the effect of temperature as given by Eq. (10); direct simulation gave results identical within statistical error.

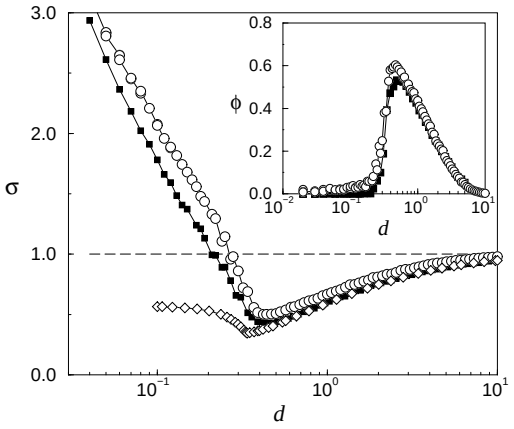

FIGURE 1. Volatility σ as a function of the reduced dimension $d = D/N$. Squares correspond to the original dynamics of Eq. (1), circles to Eq. (7); $T = 0$ and $p_i(0) \sim 0$. Diamonds correspond to minimization of $\overline{\mathcal{H}}$. Inset: fraction of frozen agents.

In [6] $\overline{\langle \mathcal{H} \rangle}$ was evaluated for $T = 0$ on the assumption that the system equilibrated and hence was equivalent to minimizing $\overline{\mathcal{H}}$. The result is also exhibited in Fig. 1 and can be seen to be good (and probably correct) for $d = D/N > d_c$ but in error for $d < d_c$. In fact, however, Eq. (7) does not describe a simple descent dynamics since the variables on the right and left hand sides are different and a metric is needed to relate p and s. Substitution shows that the dynamics in non-Markovian in s. An explicit demonstration of non-equilibration for $d < d_c$ follows from a simulation starting with $|p_i(0)| \sim \mathcal{O}(1) \gg dt$, where dt is the time-step. This is illustrated in Fig. 3a [5].

It is also tempting to compare with a Hopfield neural network which is characterizable by an effective Hamiltonian

$$H = -\frac{1}{2}\sum_{i\neq j} J_{ij}^H \sigma_i \sigma_j; \quad J_{ij}^H = N^{-1}\sum_{\mu=1}^{p=\alpha N} \vec{\xi}_i^{\mu} \cdot \vec{\xi}_j^{\mu} \tag{14}$$

where the $\{\xi_i^\mu\}$ are quenched random patterns; indeed it was by the application of techniques devised for (14) that [6] minimized $\overline{\mathcal{H}}$. Clearly there is a difference of sign between Eqs. (8) and (14) but one might be tempted to anticipate that this will merely suppress the retrieval attractors while maintaining the spin glass state, in analogy with the SK model, and then attribute the reduction in energy compared with the random state to spin-glass binding. However, this is false; the Hopfield spin-glass solution is not symmetric under change of sign of β. Rather, the random-field term of Eq. (8) is crucial in reducing the ground state energy of $\mathcal{H}$ and the volatility below their random-state values. This is demonstrated explicitly in Fig. 3b which shows the effect of choosing $\vec{R}_i^1$ randomly but $\vec{R}_i^2$ as

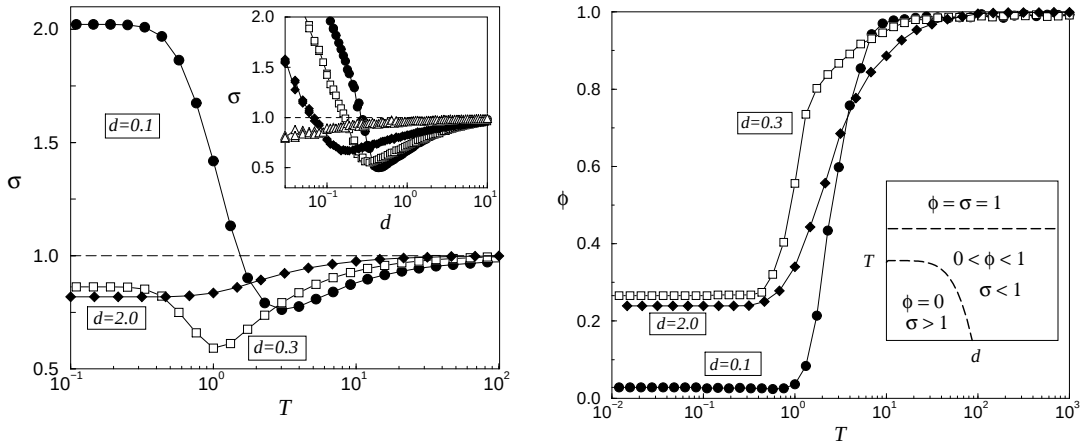

FIGURE 2. (a) Volatility as a function of the temperature from the continuous dynamics Eq. (10);$f(p) = \mathrm{sgn}(p)$, $p_i(0) \sim 0$. Inset: volatility as a function of d for $T = 10^{-3}$, 1, 2, and 10. (b) Fraction of frozen agents as a function of T. Inset: schematic cross-over phase diagram.

$$\vec{R}_i^2 = -(1-\lambda)\vec{R}_i^1 + \lambda\vec{\tilde{R}}_i \tag{15}$$

where $\vec{\tilde{R}}_i$ is also chosen randomly [10,11]. For $\lambda = 0, \vec{\omega}_i = 0, h_i = 0$ and the volatility never falls below random, although again there appears a (different) critical d_c separating regimes (worse-than-random and random).

As noted, above we have used $f(p) = \mathrm{sgn}(p)$ in the simulations. If instead $f(p) = p$ is employed, then for $d > d_c$ the system iterates over a long time to its zero-temperature behaviour [7] since the mean $|p_i(t)|$ grows quasi-continuously and $s_i(t)$ saturates to its zero-temperature value, which being ± 1 eliminates the effects of the second term of Eq. (10). However, for $d < d_c$ there continues to be an improvement with temperature, to an optimal value which is better than random and is reached at a temperature of $\mathcal{O}(1)$ [8], but without any further rise to the random value; in this case $p_i(t)$ fluctuates around $p_i(0)$.

Finally we remark on the relationship with the crowd-anticrowd concept of [9], where a crowd is a group of agents playing the same strategy and the corresponding anticrowd play the opposite strategy. From Eq. (6)

$$\sigma^2 = D^{-1}\sum_\mu \langle n_\mu(t)\rangle^2; \quad n_\mu = N^{-1/2}\sum_i \vec{R}_i^*(t)\cdot\vec{e}_\mu \tag{16}$$

where $\vec{e}_\mu$ is a unit vector in the μ^{th} Cartesian direction of the D-dimensional space. n_μ then formalizes the notion of the number of agents in crowd μ minus the number in the corresponding anti-crowd. The qualitative difference of $d < d_c$ and $d > d_c$ then follows from the recognition that for $d \ll d_c$ the vectors $\vec{R}_i^\alpha$ are densely distributed on the D-sphere permitting $n_\nu \sim \mathcal{O}(N)$, while for $d \gg d_c$ they are sparsely distributed so all $n_\nu \sim \mathcal{O}(1)$.

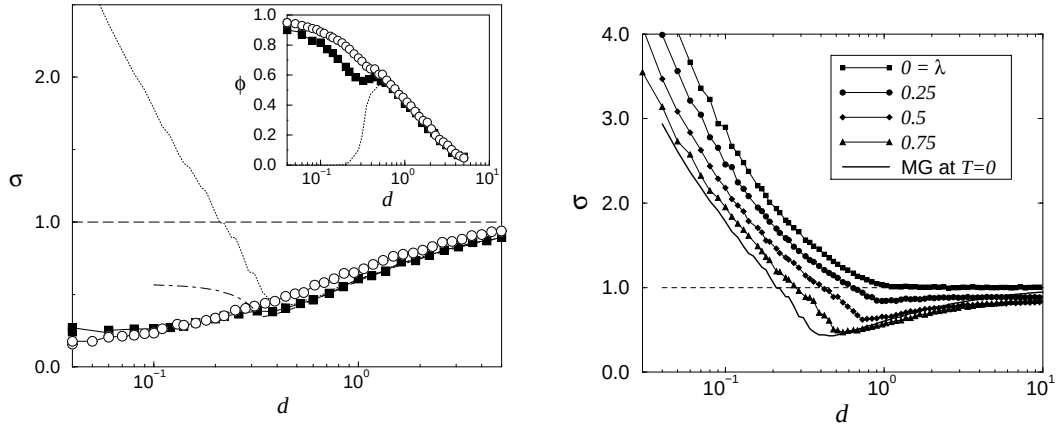

FIGURE 3. (a) Volatility as a function of d for random initial conditions $|p_i(0)| \sim \mathcal{O}(1) \gg dt$ for original dynamics and from Eq. (10). (b) Volatility for partially anticorrelated $\vec{R}_i^{1,2}$; $\vec{R}_i^2 = -(1-\lambda)\vec{R}_i^1 + \lambda\vec{\tilde{R}}_i$; $\vec{R}_i^1, \vec{\tilde{R}}_i$ random.

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